

B. Scpt III (H), Paper-V
Schwartz's Theorem

Theorem: Let (a, b) be a point of the domain of a function f such that

- (i) f_x exists in a certain neighbourhood of (a, b) .
- (ii) f_y is continuous at (a, b) .

Then f_{yx} at (a, b) exists and is equal to f_{xy} at (a, b) .

Proof:

The given condition imply that there exists a certain nbd. of (a, b) at every point (x, y) of which $f_x(x, y)$, $f_y(x, y)$ and $f_{xy}(x, y)$ all exist.

Let $(a+h, b+k)$ be any point of the nbd.

We write

$$F(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$$

$$g(y) = f(a+h, y) - f(a, y)$$

So that $F(h, k) = g(b+k) - g(b)$ (1)

Since f_y exists in a nbd. of (a, b) , the function g of one variable y is derivable in $[b, b+k]$ and therefore by applying the mean-value theorem to the expression on the right of (1), we have

$$F(h, k) = kg'(b+\theta k) \text{ where } 0 < \theta < 1$$

$$= k [f_{xy}(a+h, b+\theta k) - f_{xy}(a, b+\theta k)] \quad (2)$$

Again since f_{xy} exists in a nbd. of (a, b) , therefore by applying the mean-value theorem to the right of (2), we get

$$F(h, k) = k [h f_{xy}(a+\theta'h, b+\theta k) \text{ where } 0 < \theta' < 1$$

$$\Rightarrow \frac{F(h, k)}{hk} = f_{xy}(a+\theta'h, b+\theta k)$$

$$\Rightarrow \frac{1}{k} \left[\frac{f(a+\theta'h, b+k) - f(a, b+k)}{h} - \frac{f(a+h, b) - f(a, b)}{h} \right]$$

$$= f_{xy}(a+\theta'h, b+\theta k)$$

Since f_x exists in a nbd. of (a, b) , therefore on taking the limit as $h \rightarrow 0$, we get

$$\frac{1}{k} [f_x(a, b+k) - f_x(a, b)] = \lim_{h \rightarrow 0} f_{xy}(a+\theta'h, b+\theta k)$$

Now, let $k > 0$. Then

$$\lim_{k \rightarrow 0} \frac{f_x(a, b+k) - f_x(a, b)}{k}$$

$$= \lim_{k \rightarrow 0} \lim_{h \rightarrow 0} f_{xy} (a + \delta h, b + \epsilon k)$$

$$\Rightarrow f_{yx}(a, b) = f_{xy}(a, b),$$

Since f_{xy} is continuous at (a, b)

This proves the theorem.